

Toward a definition of the Quantum Ergodic Hierarchy: Kolmogorov and Bernoulli systems

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Abstract

In this paper we translate the two higher levels of the Ergodic Hierarchy, the Kolmogorov level and the Bernoulli level, to quantum language. This paper can be considered as the second part of the paper [1] to complete the quantum language translation of all levels of the Hierarchy Ergodic. As in paper [1], we consider the formalism where the states are positive functionals on the algebra of observables and we use the properties of the Wigner transform [7]. Further conclusions are analyzed on the validity of this approach as a formal treatment of quantum chaos.

Key words: Ergodic-Mixing-Kolmogorov-Bernoulli-UEH-QUEH

1 Introduction

In paper [1] we have studied the definition of quantum chaos using as a base a reliable definition of the classical to quantum limit since “the real conflict, which pose a threat to the correspondence principle, would arise only if any classical limit of quantum systems would not display chaotic behavior” (see introduction of paper [1], pages 218 to 248, for details).

So in paper [1], that we consider as the *first part* of this paper, we have defined the quantum chaos in the two first levels of the ergodic hierarchy (EH). ergodic and mixing. In this paper we will complete the work adding two more levels: Kolmogorov and Bernoulli. Then this paper can be considered as a *second part* of paper [1]. Nevertheless in this paper following the ideas of paper [2] we will first repeat the two initial levels using these new concepts and then add the two final levels. Also, for the sake of conciseness, we will not repeat the following sections of paper [1]: section 2 (Mathematical background), section 3 (Decoherence in non integral systems), section 4 (The classical statistical limit), and section 5 (The classical limit). These sections can be reeded in [1]. The just quoted section 5 can be also complemented with paper [3].

The paper is organized as follows: Section 2: We present the formalism, definitions and the Uniform Ergodic Hierarchy (UEH) we will use. Section 3 and 4: we briefly review the ergodic and mixing systems already considered in ref. [1]. Section 5 and 6: We explain in detail the Kolmogorov and Bernoulli cases. Section 7: We consider the relevance of the subject and draw our conclusions.

2 Formalism

For a Hilbert space \mathcal{H} of dimension $N > 2$ the set of pure states forms a $(2N - 2)$ -dimensional manifold, of measure zero, in the $(N^2 - 2)$ -dimensional boundary $\partial\mathcal{C}_N$ of the set \mathcal{C}_N of density matrices. The set of mixed quantum states \mathcal{C}_N consists of Hermitian, positive matrices of size N , normalized by the trace condition, that is

$$\mathcal{C}_N = \{\rho : \rho = \rho^\dagger; \rho \geq 0; \text{tr}(\rho) = 1; \dim(\rho) = N\}. \quad (1)$$

It can be shown for finite dimensional bipartite states that there exist always a non-zero measure μ_s in the neighborhood of separable states containing maximum uncertainty ones. μ_s tends to zero as the dimension tends to infinity. Finally, for an infinitely dimensional Hilbert space almost all states are entangled [11, 12].

In this section we would like to establish the relation of three different levels:

- The notion of *sets correlations*, the main tool of paper [2].
- The *algebra* of the observables and states (symbolized by density or distribution functions) in the phase space of the Hamiltonian Mechanics (see [4] and [5]).
- The same algebra at *quantum mechanics level* obtained through the Weyl-Wigner-Moyal transformation from the Hamiltonian Mechanics (see [7] and [8] for details).

2.1 Definitions

- Let $[X, \Sigma, \mu, T_t]$ be a generic *dynamical system*, where X is a set, Σ is a σ -algebra, μ is a measure, and T_t is a time transformation. In Hamiltonian Mechanics, the physical case, X is the phase space with coordinates $\phi = (q, p)$, (or a projection ΠX of phase space whose coordinates will also symbolized ϕ), Σ is the σ -algebra of measurable sets of X , μ is the Liouville measure $d\phi = dqdp$, (usually normalized as $\mu(X) = 1$), and T_t is given by the Hamiltonian dynamics.
- The Σ algebra has the following properties:
 - (1) $X \in \Sigma$,
 - (2) $A \setminus B \in \Sigma$ for all $A, B \in \Sigma$, and
 - (3) $\bigcup_{i=1}^n B_i \in \Sigma$ if $B_i \in \Sigma$ for $1 \leq i \leq n \leq \infty$. As a consequence Σ also contains \emptyset and $\bigcap_{i=1}^n B_i$ if $B_i \in \Sigma$, for $1 \leq i \leq n \leq \infty$
- The probability measure μ on Σ such that
 - (1) $\mu : \Sigma \rightarrow [0, 1]$ with $\mu(X) = 1$, and
 - (2) If $\{B_i\}_{i=1}^n \subset \Sigma$ and $B_j \cap B_k = \emptyset$ for $1 \leq j < k \leq n$ then $\mu(\bigcup_{i=1}^n B_i) = \sum_{i=1}^n \mu(B_i)$.

- The automorphism T is an automorphism that maps the probability space $[X, B, \mu]$ onto itself and it is measure preserving iff $B \in X$ i.e.:
 - (1) $T^{-1}B \in \Sigma$,
 - (2) $\mu(T^{-1}B) = \mu(B)$, where $T^{-1}B = [x \in X : Tx \in B]$
- A dynamical law or time evolution $\tau = \{T_t\}_{t \in I}$ is a group of measure preserving automorphisms $T_t X \rightarrow X$ of the probability space $[X, B, \mu]$ onto itself and where I is either \mathbb{R} or \mathbb{Z} .
- The set $\alpha = \{\alpha_i : i = 1, \dots, N\}$ is a partition of X iff
 - (1) $\alpha_i \cap \alpha_j = \emptyset$ for all $i \neq j$,
 - (2) $\mu(X \setminus \bigcup_{i=1}^N \alpha_i) = 0$.
 - (3) Given two partitions $\alpha = \{\alpha_i : i = 1, \dots, N\}$, $\beta = \{\beta_j : j = 1, \dots, M\}$ we will call their sum to $\alpha \vee \beta = \{\alpha_i \cap \beta_j : i = 1, \dots, N; j = 1, \dots, M\}$
- A σ -sub algebra $\Sigma_0 \subset \Sigma$ must also satisfy the conditions
 - (1) $\Sigma_0 \subseteq T\Sigma_0$,
 - (2) $\bigvee_{n=-\infty}^{\infty} T^n \Sigma_0 = \Sigma$,
 - (3) $\bigwedge_{n=-\infty}^{\infty} T^n \Sigma_0 = N$ namely the σ -algebra containing the set of measure one and zero¹.
- Let A and B be measurable sets of the space X , and let μ be the measure just defined. Then the *correlation* between A and B is defined as

$$C(B, A) = \mu(A \cap B) - \mu(A)\mu(B) \quad (2)$$

Let us explain the meaning of this notion. In a generic system and under generic circumstances we have $C(B, A) \neq 0$. But if $C(B, A) = 0$ some kind of “homogeneity” has appear in the system since both factors $\mu(A)$ and $\mu(B)$ play the same role in the product $\mu(A \cap B)$, precisely:

$$\mu(A \cap B) = \mu(A)\mu(B) \quad (3)$$

This “homogenization” in the behavior of $\mu(A \cap B)$ corresponds to the vanishing of correlations. Then if the time evolution T_t conserve the measures or $\mu(T_t A) = \mu(A)$, (as in the phase space) we have

$$\mu(T_t A \cap B) = \mu(A)\mu(B) + C(B, T_t A) \quad (4)$$

where $\mu(A)\mu(B)$ would be the “homogenous” constant part of $\mu(T_t A \cap B)$ and $C(B, T_t A)$ the “non-homogenous” variable part. Then if, e. g., when $t \rightarrow \infty$ we have $C(B, A) \rightarrow 0$ some homogenization has take place in the system².

¹Of course, the last condition cannot be fulfilled in the quantum case because the phase space will have a intrinsic graininess originated in the uncertainty principle. So we must consider that this condition would be only approximately satisfy.

²Correlations are also related with the notion of *unpredictability* [2], but we do not consider this subject in this paper.

2.2 Uniform Ergodic Hierarchy (UEH)

Using the notion of *correlation* (see equations (2) and (4)) we can define the main four steps of the UEH as:

- *Uniformly Ergodic* systems if

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T C(T_t B, A) dt = 0 \quad (5)$$

We will call this limit a *Cesàro* limit.

- *Uniformly Mixing* system if

$$\lim_{t \rightarrow \infty} C(T_t B, A) = 0 \quad (6)$$

We will call this limit a *weak* limit.

- *Uniformly Kolmogorov* systems. Using the Cornfeld-Fomin-Sinai theorem ([9] page 283) the traditional definition for these system can be translated to the correlations language into following:

A system is uniformly Kolmogorov if for any integer r and any set $A_0, A_1, A_2, \dots, A_r \in X$ and for any $\varepsilon > 0$ there exists an $n_0 > 0$ such for all $B \in \sigma_{n,r}(A_1, A_2, \dots, A_r)$, we have

$$|C(B, A_0)| < \varepsilon \quad (7)$$

where $\sigma_{n,r}(A_1, A_2, \dots, A_r)$ is a sub σ -algebra (defined in section 2.1.vii)

For example this σ -algebra contains, among others, the following sets.

1. All the $T_k A_i$, for all $k \geq n$, and all $i = 1, \dots, r$.
 2. All the finite and infinite sequences $T_n A_{m_1} \cup T_{n+1} A_{m_2} \cup T_{n+2} A_{m_3} \dots$, and $T_n A_{m_1} \cup T_{n+1} A_{m_2} \cup T_{n+2} A_{m_3} \dots$ where $m_i \in (i = 1, \dots, r)$
 3. All the finite and infinite sequences $T_n A_{m_1} \cap T_{n+1} A_{m_2} \cap T_{n+2} A_{m_3} \dots$ and $T_n A_{m_1} \cap T_{n+1} A_{m_2} \cap T_{n+2} A_{m_3} \dots$ where $m_i \in (i = 1, \dots, r)$
- *Uniformly Bernoulli* system if for any time t

$$C(T_t B, A) = 0 \quad (8)$$

so from eq. (4) if $\mu(T_t A) = \mu(A)$

$$\mu(T_t B \cap A) = \mu(A)\mu(B) \quad (9)$$

i.e. in probability language: the probability to obtain the event B , at any time, conditioned by A is always the same and we have the homogeneity defined in eq. (3).

Then the levels of the UEH are defined *by the way the correlations vanishes* when $t \rightarrow \infty$ (being the Bernoulli level defined by a trivial zero identity).

2.3 Correlations at the different levels

Now we can also define the notion of correlation at their different levels of subsection 2.2.

I) Measurable set level

$$C(B, A) = \mu(A \cap B) - \mu(A)\mu(B) \quad (10)$$

II) Distribution or density function level

$$C(g, f) = \langle f, g \rangle - \langle f, 1 \rangle \langle 1, g \rangle \quad (11)$$

where f (and g) is a function over the phase space X such that the integral $\int_X f(\phi) d\phi$ exists, $\langle f, g \rangle = \int_X f(\phi) g(\phi) d\phi$ and where $\phi = (q, p)$, are the coordinates at a point of X , so $\phi \in X$ and $d\phi = \mu(d\phi) = dqdp$.

III) Quantum level

$$C(\widehat{g}|\widehat{f}) = (\widehat{f}|\widehat{g}) - (\widehat{f}|\widehat{I})(\widehat{I}|\widehat{g}) \quad (12)$$

where and \mathcal{A} is the algebra of observables. Then if $(\widehat{f}|$ and $g = \text{symb}(\widehat{f}|$ are the Weyl-Wigner-Moyal transforms of $(\widehat{f}|$ and $|\widehat{g})$ we know that $(\widehat{f}|\widehat{g}) = \langle f, g \rangle^3$. Then using the usual quantum symbols for observables and states and we have

$$C(\widehat{O}, \widehat{\rho}) = (\widehat{\rho}|\widehat{O}) - (\widehat{\rho}|\widehat{I})(\widehat{I}|\widehat{O}) \quad (13)$$

where $\widehat{\rho} = \widehat{f}$ are the states and $\widehat{O} = \widehat{g}$ are the observables.⁴

From these equations we can see that we can translate the UEH up to a quantum uniform ergodic hierarchy (QUEH), has we have done for the two first steps, for the ergodic hierarchy, in paper [1].

Let us now schematically show the relations among eqs., (10) to (13). Let us define the characteristic function $1_A(\phi)$ as

$$1_A(\phi) = 1 \text{ if } \phi \in A, \quad 1_A(\phi) = 0 \text{ if } \phi \notin A$$

Then as $1_A^2(\phi) = 1_A(\phi)$, and $1_A(\phi)$ can also be considered as a projector $\Pi_A(\phi) = 1_A(\phi)$. Using these projectors we can write definition (10) as

$$C(B, A) = \int_X 1_A(\phi) 1_B(\phi) d\phi - \int_X 1_A(\phi) d\phi \int_X 1_B(\phi) d\phi \quad (14)$$

since it is evident that the terms of the r.h.s. of both equations are the same.

Let us now define a partition $\{A_i\}$ of X that satisfies

$$X = \bigcup_i A_i, \quad A_i \cap A_j = \emptyset \text{ if } i \neq j$$

or such that

$$1_{A_i} 1_{A_j} = \delta_{ij} 1_{A_i}$$

³In the process, from *I*) to *III*), we may say that the ignorance probabilities become intrinsic probabilities, but numerically they are equal and they predict in the same way.

⁴The normalization of $\widehat{\rho}(t)$ is simply $(\widehat{\rho}|\widehat{I}) = 1$ or $\text{Tr}\widehat{\rho} = 1$ so $C(\widehat{O}, \widehat{\rho}) = (\widehat{\rho}|\widehat{O}) - (\widehat{I}|\widehat{O}) = (\widehat{\rho}|\widehat{O}) - \text{Tr}\widehat{O}$

Let us also introduce two arbitrary sets of number $a_i, b_j \in \mathbb{R}$, then from eq. (14)

$$\begin{aligned} & \sum_{ij} a_i b_j C(A_j, A_i) = \\ &= \sum_{ij} \int_X a_i b_j 1_{A_i}(\phi) 1_{A_j}(\phi) d\phi - \sum_i \int_X a_i 1_{A_i}(\phi) d\phi \sum_j \int_X b_j 1_{A_j}(\phi) d\phi \end{aligned} \quad (15)$$

Then if we define two functions

$$f(\phi) = \sum_i a_i 1_{A_i}(\phi), \quad g(\phi) = \sum_j b_j 1_{A_j}(\phi)$$

it is clear that since we can make the domains A_i of the partition as small as we want we can approximate any possible functions $f(\phi), g(\phi)$, then we can define

$$C(g, f) = \sum_{ij} a_i b_j C(A_j, A_i) = \int_X f(\phi) g(\phi) d\phi - \int_X f(\phi) d\phi \int_X g(\phi) d\phi$$

or defining $\langle f(\phi), g(\phi) \rangle = \int_X f(\phi) g(\phi) d\phi$.

$$C(g, f) = \sum_{ij} a_i b_j C(A_j, A_i) = \langle f(\phi), g(\phi) \rangle - \langle f(\phi), 1 \rangle \langle 1, g(\phi) \rangle \quad (16)$$

i.e. the definition of correlations but now in the *distribution function language* (cf. eq. (11)) is demonstrated. This definition is equivalent to (10) if $a_i = \delta_{i0}, b_j = \delta_{j1}, A_0 = B, A_1 = A$.

Given that $\langle \hat{\rho} | \hat{I} \rangle = \langle \rho, 1 \rangle = \langle \rho \rangle$ and $\text{symp} \hat{I} = 1$, applying to (16) the Weyl-Wigner-Moyal transform and interpreting \hat{f} as the state and \hat{g} as the operator, if $\text{symp} \hat{O} = O(\phi)$ and $\text{symp} \hat{\rho} = \rho(\phi)$, we have that

$$C(\hat{O}, \hat{\rho}) = \langle \hat{\rho} | \hat{O} \rangle - \langle \hat{\rho} | \hat{I} \rangle \langle \hat{I} | \hat{O} \rangle \quad (17)$$

i.e. the definition of correlations but now at the *quantum language* (cf. eq. (13)) which is equivalent to (16) from the properties of Weyl-Wigner-Moyal transform. So when $\hbar \rightarrow 0$ we have $(17) \Leftrightarrow (16) \Leftrightarrow (10)$.

So we can see that the three levels: measurable set level, distribution function level, and quantum level are all equivalent and interchangeable.

2.4 More general equations and the Ergodic Hierarchy (EH)

- We will call the evolution operator of distribution or density function is the Frobenius-Perron operator P_t . In quantum language the Frobenius-Perron operator P_t would be the evolution operator for states, while the Koopman operator U_t would be the time evolution operator for observables. In fact we have that

$$\langle P_t f, g \rangle = \langle f, U_t g \rangle \quad (18)$$

see [4] eq.(3.3.4).

Then P_t , the Frobenius-Perron operator, conserve the measure. Then we have $\int_X P_t 1_{A_i} d\phi = \int_X 1_{A_i} d\phi$ and $\sum_i a_i \int_X P_t 1_{A_i} d\phi = \sum_i a_i \int_X 1_{A_i} d\phi$ thus

$$\int_X P_t f d\phi = \int_X f d\phi \text{ or } \langle P_t f \rangle = \langle f \rangle \quad (19)$$

or at the quantum level, since $\langle f \rangle = \langle f \cdot I \rangle = (\hat{f} | \hat{I}) = Tr \hat{f}$, we have

$$Tr(\hat{\rho}(t)) = Tr(\hat{\rho}(0)) \quad (20)$$

namely the trace is also conserved.

- In general there it may exist several f_* , the *equilibrium distributions* such, that $P_t f_* = f_*$. But if the systems is ergodic there is only one of them, therefore we will consider only this case.
- At the two first levels of the EH we will have a limit (Cesàro, Mixing) $P_t f \rightarrow f_*$ when $t \rightarrow \infty$ and from these limit we will have $\langle f_* \rangle = \langle f(t) \rangle$ or $Tr \hat{\rho}_* = Tr \rho(t)$, since the norm is also conserve at the limit.

Then we can define a *new measure* $\mu_*(A)$ such that

$$\mu_*(A) = \int_A f_*(\phi) d\phi$$

and define a new correlation

$$C_*(B, A) = \mu_*(A \cap B) - \mu_*(A)\mu_*(B)$$

Now we can define the new levels: *Ergodic and Mixing* making $\mu \rightarrow \mu_*$ in eqs. (5) to (8). So we have the UEH and at the beginning of (simply) Ergodic Hierarchy (EH)⁵.

Then, e. g., in the mixing case, we have, e.g. for the mixing case (see [6] pag. 58)

$$\lim_{t \rightarrow \infty} \mu_*(T_t A \cap B) = \mu_*(A)\mu_*(B) \quad (21)$$

then

$$\lim_{t \rightarrow \infty} \mu_*(T_t A \cap X) = \mu_*(A)\mu_*(X)$$

and if we normalize $\mu_*(X) = 1$.

$$\lim_{t \rightarrow \infty} \mu_*(T_t A) = \mu_*(A)$$

i.e. the conservation of the normalization is also valid at the limit $t \rightarrow \infty$, (an example of point iii above).

⁵This difference is only possible if in X we have a measure μ that can be consider as uniform, as in phase space, or the Sinai-Ruelle-Bowen measure in generic hyperbolic systems, but not in the generic case.

- Let us quote the Theorem 5.1 of [6]:

“Let T_t be an ergodic transformation, with stationary density $f_*(\phi)$ of the associated Frobenius-Perron operator, operating in a phase space of finite μ_* measure. Then T_t is mixing iff $\{P_t f\}$ is weakly convergent to $f_*(\phi)$ for all densities f , i. e.

$$\lim_{t \rightarrow \infty} \langle P_t f, g \rangle = \langle f_*, g \rangle$$

for every bounded measurable function g ”.

The demonstration is as follows:

$$\begin{aligned} \lim_{t \rightarrow \infty} \mu_*(T_t A \cap B) &= \lim_{t \rightarrow \infty} \int_{T_t A \cap B} f_*(\phi) d\phi = \lim_{t \rightarrow \infty} \int_X 1_{T_t A \cap B} f_*(\phi) d\phi = \\ &= \lim_{t \rightarrow \infty} \int_X 1_{T_t A} 1_B f_*(\phi) d\phi = \lim_{t \rightarrow \infty} \langle P_t 1_A f_*(\phi), 1_B \rangle \end{aligned} \quad (22)$$

and also

$$\mu_*(A) \mu_*(B) = \int_X 1_A f_*(\phi) d\phi \int_X 1_B f_*(\phi) d\phi = \langle 1_A f_*(\phi), 1 \rangle \langle f_*(\phi), 1_B \rangle$$

so from eq. (21) we have

$$\lim_{t \rightarrow \infty} \langle P_t 1_A f_*(\phi), 1_B \rangle = \langle 1_A f_*(\phi), 1 \rangle \langle f_*(\phi), 1_B \rangle$$

or

$$\lim_{t \rightarrow \infty} \langle P_t 1_{A_i} f_*(\phi), 1_{B_j} \rangle = \langle 1_{A_i} f_*(\phi), 1 \rangle \langle f_*(\phi), 1_{B_j} \rangle$$

so considering two set of generic numbers (a_i) and (b_j) and define the generic functions

$$f = \sum_i a_i 1_{A_i} f_*(\phi), \quad g = \sum_j b_j 1_{B_j},$$

we obtain

$$\lim_{t \rightarrow \infty} \langle f, g \rangle = \langle f, 1 \rangle \langle f_*(\phi), g \rangle$$

and if f is normalized as $\langle f, 1 \rangle = 1$ the thesis follows. q.e.d.

Or in other words,

$$W - \lim_{t \rightarrow \infty} P_t f = f_* \quad (23)$$

Finally the corresponding definition of quantum mixing is

$$\lim_{t \rightarrow \infty} (\hat{\rho}(t) | \hat{O}) = (\hat{\rho}_* | \hat{O}) \quad (24)$$

namely $\hat{\rho}(t)$ weakly converges to $\hat{\rho}_*$ (see [1]).

For the *ergodic* case we must simply make the substitution $\lim_{t \rightarrow \infty} \rightarrow \lim_{t \rightarrow \infty} \frac{1}{T} \int_0^T$ or $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_0^{n-1}$ in the discrete case. The Kolmogorov and Bernoulli cases will be considered in sections 5 and 6.

In Table I we display the synthetic structure of the three levels

TABLE I (*SET LEVEL, DISTRIBUTION FUNCTION LEVEL, QUANTUM LEVEL*)

	SETS	FUNCTIONS	QUANTUM OPERATORS
EVOLUTION (projectors)	$A \rightarrow TA$ Liouville ev.	$1_A \rightarrow P_t 1_A = 1_{T_A}$ Frob.-Perron ev.	$\text{symp}^{-1} 1_{T_A} = \hat{P}_A(t)$ Heisenberg ev.
EQUILIBRIUM (states)		$U_t f_* = f_*$ Koopman ev.	$\hat{U}_t \hat{\rho}_* \hat{U}_t^\dagger = \hat{\rho}_*$ Schroedinger ev.
OPERATIONS	$A \cap B$ $A \cup B$	$1_A 1_B$ $1_A + 1_B - 1_{A \cap B}$	$\hat{P}_A \hat{P}_B, \hbar \sim 0$ $\hat{P}_A + \hat{P}_B - \hat{P}_{A \cap B}, \hbar \sim 0$

3 Ergodic Systems

According to paper [2] eq. (E) the system is uniform ergodic if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T_k B \cap A) = \mu(A)\mu(B) \quad (25)$$

or

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} C(T_k B, A) = 0 \quad (26)$$

But if we introduce the measure $\mu_*(A)$, as we have define the new *Ergodic level* making $\mu \rightarrow \mu_*$ we have that the system is ergodic if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu_*(T_k A \cap B) = \mu_*(A)\mu_*(B) \quad (27)$$

or for the distribution of density function case (see also the corresponding theorem 4.7 in [6]) or in the continuous case

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle P_t f, g \rangle dt = \langle f_*, g \rangle$$

or finally in the quantum case, it is quantum ergodic if

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\hat{\rho}(t) | \hat{O}) dt = (\hat{\rho}_* | \hat{O}) \quad (28)$$

as explained in all detail in the first part of this paper i.e. [1].

4 Mixing Systems

According to paper [2] eq. (M) the system is *uniform mixing* if

$$\lim_{n \rightarrow \infty} \mu(T_n B \cap A) = \mu(A)\mu(B) \quad (29)$$

or

$$\lim_{n \rightarrow \infty} C(T_n B, A) = 0 \quad (30)$$

Moreover from [6] p. 58 system is *mixing* if

$$\lim_{n \rightarrow \infty} \mu_*(T_n A \cap B) = \mu_*(A)\mu_*(B) \quad (31)$$

or for the distribution of density function case (see also the corresponding theorem 5.1 in [6])

$$\langle P_t f, g \rangle = \langle f_*, g \rangle$$

or in the quantum case, it is quantum mixing if

$$\lim_{t \rightarrow \infty} (\widehat{\rho}(t)|\widehat{O}) = (\widehat{\rho}_*|\widehat{O}) \quad (32)$$

as explained in all detail in the first part i.e. [1].

5 Kolmogorov Systems

The two previous sections are essentially contained in [1] and they were introduced here for the sake of completeness. This section is the main part of the paper. We remark that things are not so simple at the Kolmogorov level essentially because the theorem in section 2.4 cannot be reproduced. We begin by recalling the definition of Kolmogorov systems at the measurable set level.

5.1 Kolmogorov systems in the UEH

We return to the definition of uniform Kolmogorov system of subsection 2.2:

A system is uniformly Kolmogorov if for any integer r and any set $A_0, A_1, A_2, \dots, A_r \in X$ and for any $\varepsilon > 0$ there exists an $n_0 > 0$ such for all $B \in \sigma_{n,r}(A_1, A_2, \dots, A_r)$ and any $n > n_0$ we have

$$|C(B, A_0)| < \varepsilon \quad (33)$$

That is,

$$\lim_{n \rightarrow \infty} C(B, A_0) = \lim_{n \rightarrow \infty} \{\mu(B \cap A_0) - \mu(B)\mu(A_0)\} = 0 \quad \forall B \in \sigma_{n,r}(A_1, A_2, \dots, A_r) \quad (34)$$

where $\sigma_{n,r}(A_1, A_2, \dots, A_r)$ is the σ -algebra generated by $\{T^k A_i : k \geq n ; i = 1, \dots, r\}$, and therefore $\sigma_{n,r}(A_1, A_2, \dots, A_r) = \sigma(\{T^k A_i : k \geq n ; i = 1, \dots, r\})$

Recall that if f_* is an stationary density, namely $P_t f_* = f_*$ then the measure μ_* given by

$$\mu_*(A) = \int_A f_*(\phi) d\phi \quad \forall A \in X \quad (35)$$

is an invariant measure (i.e. $\mu_*(S^{-1}(A)) = \mu_*(A)$ for all transformation $S : X \rightarrow X$ and for all $A \in X$) (see Theorem 4.1.1. of [4]).

As we consider the previous sections, we make $\mu = \mu_*$ and therefore the Kolmogorov condition (34) becomes

$$\lim_{n \rightarrow \infty} \{\mu_*(B \cap A_0) - \mu_*(B)\mu_*(A_0)\} = 0 \quad \forall B \in \sigma_{n,r}(A_1, A_2, \dots, A_r) \quad (36)$$

Now a question arises, What are the sets containing the σ -algebra $\sigma(\{T^k A_i : k \geq n ; i = 1, \dots, r\})$? There are two types of these sets:

$$(I) \quad B = \bigcup_i T_{n+n_i} A_{s_i} \setminus T_{n+l_i} A_{p_i} \quad (\text{finite or countable unions of } T_i A_i \setminus T_j A_j)$$

$$(II) \quad B = \bigcap_i T_{n+n_i} A_{s_i} \quad (\text{finite or countable intersections of } T_i A_i)$$

where $n_i, l_i \in \mathbb{N}_0$ y $s_i, p_i \in \{1, \dots, r\}$.

It is clear that (finite or countable) unions of $T_i A_i$ are included because it is sufficient to make in (I) $A_{p_i} = \emptyset$ for all p_i and results $B = \bigcup T_{n+n_i} A_{s_i}$.

Therefore, if we can translate the condition (36) into quantum language for the sets of type (I) and (II) we will have the Kolmogorov Quantum Hierarchy in the UEH. We begin with the sets of type (I):

We have that for these type of sets the condition (36) becomes

$$\lim_{n \rightarrow \infty} \{\mu_*(\bigcup_i T_{n+n_i} A_{s_i} \setminus T_{n+l_i} A_{p_i} \cap A_0) - \mu_*(\bigcup_i T_{n+n_i} A_{s_i} \setminus T_{n+l_i} A_{p_i})\mu_*(A_0)\} \quad (37)$$

which is equals to

$$\lim_{n \rightarrow \infty} \{\mu_*(\bigcup_i T_{n+n_i} A_{s_i} \cap (T_{n+l_i} A_{p_i})^c \cap A_0) - \mu_*(\bigcup_i T_{n+n_i} A_{s_i} \cap (T_{n+l_i} A_{p_i})^c)\mu_*(A_0)\} \quad (38)$$

Now by the *inclusion-exclusion principle* (see for example [13]) if P is a measure of probability and $Z_1, Z_2, Z_3, \dots, Z_n$ are sets we have that

$$P(\bigcup_{i=1}^n Z_i) = \sum_{k=1}^n \sum_{I \subseteq \{1, \dots, n\}, \#(I)=k} (-1)^{k+1} P(\bigcap_{i \in I} Z_i) \quad (39)$$

where P is the **probability** which extended for $n \rightarrow \infty$ becomes

$$P(\bigcup_{i=1}^{\infty} Z_i) = \sum_{k=1}^{\infty} \sum_{I \subseteq \mathbb{N}, \#(I)=k} (-1)^{k+1} P(\bigcap_{i \in I} Z_i) \quad (40)$$

Since that f_* is a density, more precisely $f_* \in D(X, \Sigma, \mu) = \{f \in L^1(X, \Sigma, \mu) : f \geq 0 ; \|f\| = 1\}$ (see Definition 3.1.3. of [4]), that is, $D(X, \Sigma, \mu)$ is the space of the distribution functions defined over all space phase. Then μ_* is a measure of probability and we can use (40) to express (38) like

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{I \subseteq \mathbb{N}, \#(I)=k} (-1)^{k+1} \mu_* \left(\bigcap_{j \in I} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c \cap A_0 \right) - \\
& - \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{I \subseteq \mathbb{N}, \#(I)=k} (-1)^{k+1} \mu_* \left(\bigcap_{j \in I} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c \right) \mu_*(A_0) = \\
& = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{I \subseteq \mathbb{N}, \#(I)=k} (-1)^{k+1} \{ \mu_* \left(\bigcap_{j \in I} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c \cap A_0 \right) - \\
& - \mu_* \left(\bigcap_{j \in I} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c \right) \mu_*(A_0) \} = \\
& = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{I \subseteq \mathbb{N}, \#(I)=k} (-1)^{k+1} C_* \left(\bigcap_{j \in I} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c, A_0 \right) = \\
& = \sum_{k=1}^{\infty} \sum_{I \subseteq \mathbb{N}, \#(I)=k} (-1)^{k+1} \lim_{n \rightarrow \infty} C_* \left(\bigcap_{j \in I} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c, A_0 \right) = 0
\end{aligned} \tag{41}$$

where we have used that $C_*(A, B) = \mu_*(A \cap B) - \mu_*(A)\mu_*(B)$. From the last equation (41) we see that the problem reduces to determining whether the limit

$$\lim_{n \rightarrow \infty} C_* \left(\bigcap_{j \in I} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c, A_0 \right) = 0 \tag{42}$$

exists. So if we translate (42) to quantum language the resultant condition will be the fundamental property of the quantum Kolmogorov systems (because if we make $A_{p_i} = \emptyset$ for all p_i then we obtain the condition of the sets of type (II)). In a more general way we consider infinite numerable intersections

$$\begin{aligned}
& \lim_{n \rightarrow \infty} C_* \left(\bigcap_{j=1}^{\infty} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c, A_0 \right) = \lim_{n \rightarrow \infty} \{ \mu_* \left(\bigcap_{j=1}^{\infty} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c \cap A_0 \right) - \\
& - \mu_* \left(\bigcap_{j=1}^{\infty} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c \right) \mu_*(A_0) \} = 0
\end{aligned} \tag{43}$$

Now using the definition of μ_* (see equation (35)) the equation (43) is expressed as

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left\{ \int_{\bigcap_{j=1}^{\infty} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c \cap A_0} f_*(\phi) d\phi - \right. \\
& \quad \left. - \left(\int_{\bigcap_{j=1}^{\infty} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c} f_*(\phi) d\phi \right) \left(\int_{A_0} f_*(\phi) d\phi \right) \right\} = \\
& = \lim_{n \rightarrow \infty} \left\{ \int_X 1_{\bigcap_{j=1}^{\infty} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c \cap A_0} f_*(\phi) d\phi - \right. \\
& \quad \left. - \left(\int_X 1_{\bigcap_{j=1}^{\infty} T_{n+n_j} A_{s_j} \cap (T_{n+l_j} A_{p_j})^c} f_*(\phi) d\phi \right) \left(\int_X f_*(\phi) 1_{A_0} d\phi \right) \right\} = 0
\end{aligned} \tag{44}$$

which is equals to

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left\{ \int_X \prod_{j=1}^{\infty} 1_{T_{n+n_j} A_{s_j}} (1 - 1_{T_{n+l_j} A_{p_j}}) f_*(\phi) 1_{A_0} d\phi - \right. \\
& \quad \left. - \left(\int_X \prod_{j=1}^{\infty} 1_{T_{n+n_j} A_{s_j}} (1 - 1_{T_{n+l_j} A_{p_j}}) f_*(\phi) d\phi \right) \left(\int_X f_*(\phi) 1_{A_0} d\phi \right) \right\} = 0
\end{aligned} \tag{45}$$

Moreover the characteristics functions $1_{T_{n+n_j} A_{s_j}}, 1_{T_{n+l_j} A_{p_j}}$ are equal to $P_{n+n_j} 1_{A_{s_j}}, P_{n+l_j} 1_{A_{p_j}}$ respectively. Then (45) reads as

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left\{ \int_X \prod_{j=1}^{\infty} P_{n+n_j} 1_{A_{s_j}} (1 - P_{n+l_j} 1_{A_{p_j}}) f_*(\phi) 1_{A_0} d\phi - \right. \\
& \quad \left. - \left(\int_X \prod_{j=1}^{\infty} P_{n+n_j} 1_{A_{s_j}} (1 - P_{n+l_j} 1_{A_{p_j}}) f_*(\phi) d\phi \right) \left(\int_X f_*(\phi) 1_{A_0} d\phi \right) \right\} = 0
\end{aligned} \tag{46}$$

Using that

$$1 - P_{n+l_j} 1_{A_{p_j}} = P_{n+l_j} (1 - 1_{A_{p_j}}) = P_{n+l_j} 1_{(A_{p_j})^c} \tag{47}$$

we see that the equation (46) can be expressed as

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left\{ \int_X \prod_{j=1}^{\infty} P_{n+n_j} 1_{A_{s_j}} P_{n+l_j} 1_{(A_{p_j})^c} f_*(\phi) 1_{A_0} d\phi - \right. \\
& \quad \left. - \left(\int_X \prod_{j=1}^{\infty} P_{n+n_j} 1_{A_{s_j}} P_{n+l_j} 1_{(A_{p_j})^c} f_*(\phi) d\phi \right) \left(\int_X f_*(\phi) 1_{A_0} d\phi \right) \right\} = 0
\end{aligned} \tag{48}$$

and therefore

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \{ \langle \prod_{j=1}^{\infty} P_{n+n_j} 1_{A_{s_j}} P_{n+l_j} 1_{(A_{p_j})^c} f_*(\phi), 1_{A_0} \rangle - \\
& - \langle \prod_{j=1}^{\infty} P_{n+n_j} 1_{A_{s_j}} P_{n+l_j} 1_{(A_{p_j})^c} f_*(\phi), 1 \rangle \langle f_*(\phi), 1_{A_0} \rangle \} = 0
\end{aligned} \tag{49}$$

Again here we apply the same trick used in subsection 2.4. We consider three set of generic numbers $(a_n^{(j)})$, $(b_m^{(j)})$ and (c_l) and define the generic functions

$$\begin{aligned}
f_{s_j} &= \sum_n a_n^{(j)} 1_{A_{s_j}^{(n)}} \\
f_{p_j} &= \sum_m b_m^{(j)} 1_{((A_{p_j})^c)^{(m)}} f_*(\phi) \\
g &= \sum_l c_l 1_{A_0^{(l)}}
\end{aligned} \tag{50}$$

Then we obtain that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \{ \langle \prod_{j=1}^{\infty} P_{n+n_j} f_{s_j} P_{n+l_j} f_{p_j}, g \rangle - \\
& - \langle \prod_{j=1}^{\infty} P_{n+n_j} f_{s_j} P_{n+l_j} f_{p_j}, 1 \rangle \langle f_*(\phi), g \rangle \} = 0
\end{aligned} \tag{51}$$

If we reorder the indices n_j, l_j and define the functions f_{s_j}, f_{p_j} such that $n_j = m_j, l_j = m_{j+1}$; $F_j = f_{s_j}, F_{j+1} = f_{p_j}$, that is, f_{s_j} and f_{p_j} are the F_j terms of even and odd index. We have

$$\lim_{n \rightarrow \infty} \{ \langle \prod_{j=1}^{\infty} P_{n+m_j} F_j, g \rangle - \langle \prod_{j=1}^{\infty} P_{n+m_j} F_j, 1 \rangle \langle f_*(\phi), g \rangle \} = 0 \tag{52}$$

We can rewrite (52) as

$$\lim_{n \rightarrow \infty} \{ \langle P_{n+m_1} F_1, g \prod_{j=2}^{\infty} P_{n+m_j} F_j \rangle - \langle P_{n+m_1} F_1, \prod_{j=2}^{\infty} P_{n+m_j} F_j \rangle \langle f_*(\phi), g \rangle \} = 0 \tag{53}$$

Now according to paper [1] the start product tends to product function when $\hbar \rightarrow 0$

$$f(\phi)g(\phi) \longrightarrow (f * g)(\phi) = \text{ymb}(\widehat{f}) * \text{ymb}(\widehat{g}) = \text{ymb}(\widehat{f\widehat{g}}) \tag{54}$$

and therefore when $\hbar \rightarrow 0$ for a infinite product of functions we have

$$\begin{aligned}
f_i(\phi) &= \text{ symb}(\widehat{f}_i) \\
\prod_{i=1}^{\infty} f_i(\phi) &= \prod_{i=1}^{\infty} \text{ symb}(\widehat{f}_i) = \text{ symb}(\prod_{i=1}^{\infty} \widehat{f}_i)
\end{aligned} \tag{55}$$

On the other hand the table 1 (subsection 2.4) says that

$$P_t 1_A = 1_{T_t A} = \text{ symb}(\widehat{P}_A(t)) \tag{56}$$

and therefore, if we have a generic function h

$$h = \sum_k h_k 1_{A_k} \tag{57}$$

then (see eq. (22) of paper [1] and table 1 of subsection 2.4)

$$\begin{aligned}
\text{ symb}^{-1}(1_{A_k}) &= \widehat{P}_{A_k} \\
\widehat{h} &= \text{ symb}^{-1}(h) = \sum_k h_k \text{ symb}^{-1}(1_{A_k}) = \sum_k h_k \widehat{P}_{A_k} \\
\widehat{h}(t) &= \sum_k h_k \widehat{P}_{A_k}(t) \\
P_t h &= \sum_k h_k P_t 1_{A_k} = \sum_k h_k P_t 1_{A_k} = \sum_k h_k P_t 1_{A_k} = \sum_k h_k \text{ symb}(\widehat{P}_{A_k}(t)) = \\
&= \text{ symb}(\sum_k h_k \widehat{P}_{A_k}(t)) = \text{ symb}(\widehat{h}(t))
\end{aligned} \tag{58}$$

This means that if we introduce the last equation of (58) and the equation (55) for $P_{n+m_j} F_j$ when $\hbar \rightarrow 0$ we have

$$\begin{aligned}
P_{n+m_j} F_j &= \text{ symb}(\widehat{F}_j(n+m_j)) \\
\prod_{j=2}^{\infty} P_{n+m_j} F_j &= \prod_{j=2}^{\infty} \text{ symb}(\widehat{F}_j(n+m_j)) = \text{ symb}(\prod_{j=2}^{\infty} \widehat{F}_j(n+m_j))
\end{aligned} \tag{59}$$

Now we call

$$\begin{aligned}
f_* &= \text{ symb}(\widehat{\rho}_*) \\
g &= \text{ symb}(\widehat{g})
\end{aligned} \tag{60}$$

where $\widehat{\rho}_*$ is the weak limit of $\widehat{\rho}(t)$ (see [1]).

Therefore if we use (55), (59) and (60) in (53) when $\hbar \rightarrow 0$ we have

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \{ \langle \text{ symb } (\widehat{F}_1(n + m_1)), \text{ symb } (\widehat{g} \prod_{j=2}^{\infty} \widehat{F}_j(n + m_j)) \rangle \\
& - \langle \text{ symb } (\widehat{F}_1(n + m_1)), \text{ symb } (\prod_{j=2}^{\infty} \widehat{F}_j(n + m_j)) \rangle \langle \text{ symb } (\rho_*), \text{ symb } (\widehat{g}) \rangle \} = 0
\end{aligned} \tag{61}$$

Now this equation can be expressed in the quantum level as (replacing $\langle \ , \ \rangle$ by $(\ | \)$)

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \{ (\text{ symb } (\widehat{F}_1(n + m_1)) | \text{ symb } (\widehat{g} \prod_{j=2}^{\infty} \widehat{F}_j(n + m_j))) \\
& - (\text{ symb } (\widehat{F}_1(n + m_1)) | \text{ symb } (\prod_{j=2}^{\infty} \widehat{F}_j(n + m_j))) (\text{ symb } (\rho_*) | \text{ symb } (\widehat{g})) \} = 0
\end{aligned} \tag{62}$$

At this point we rename the operators $\widehat{F}_1(n + m_1)$, $\widehat{F}_j(n + m_j)$ and \widehat{g} as $\widehat{\rho}(n + m_1)$, $\widehat{O}_j(n + m_j)$ and \widehat{O}_1 respectively. That is, we emphasize in the role of the states, and the observables and we have

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \{ (\text{ symb } (\widehat{\rho}(n + m_1)) | \text{ symb } (\widehat{O}_1 \prod_{j=2}^{\infty} \widehat{O}_j(n + m_j))) \\
& - (\text{ symb } (\widehat{\rho}(n + m_1)) | \text{ symb } (\prod_{j=2}^{\infty} \widehat{O}_j(n + m_j))) (\text{ symb } (\rho_*) | \text{ symb } (\widehat{O}_1)) \} = 0
\end{aligned} \tag{63}$$

Then using the important property that the Wigner transformation yields the correct expectation value of any observable \widehat{O} in the state $\widehat{\rho}$ (see equation (23) of paper [1]) we have

$$\lim_{n \rightarrow \infty} \{ (\widehat{\rho}(n + m_1) | \widehat{O}_1 \prod_{j=2}^{\infty} \widehat{O}_j(n + m_j)) - \prod_{j=2}^{\infty} (\widehat{\rho}(n + m_1) | \widehat{O}_j(n + m_j)) (\widehat{\rho}_* | \widehat{O}_1) \} = 0 \tag{64}$$

Finally, the definition of the quantum Kolmogorov level is

$$\lim_{n \rightarrow \infty} \{ (\widehat{\rho}(n + m_1) | \widehat{O}_1 \prod_{j=2}^{\infty} \widehat{O}_j(n + m_j)) - \prod_{j=2}^{\infty} (\widehat{\rho}(n + m_1) | \widehat{O}_j(n + m_j)) (\widehat{\rho}_* | \widehat{O}_1) \} = 0 \tag{65}$$

for all observables $\widehat{O}_2, \widehat{O}_3, \widehat{O}_4, \dots$ and all $m_1, m_2, m_3, \dots \in \mathbb{N}_0$ where $\widehat{\rho}_*$ is the weak limit of $\widehat{\rho}(t)$.

5.2 Particular Case: Mixing

According to the definition of Kolmogorov level, we know that the mixing level includes Kolmogorov level, that is, the equation (34) implies the equation (29). Therefore it would be expected for a good definition of quantum Kolmogorov level given by equation (65) that from this equation we can deduce the quantum level mixing given by equation (32). If we make $O_1 = \hat{O}$, $O_i = \hat{I}$ for all $i=2,3,4,\dots$ and $m_1 = 0$ in the equation (65) we have

$$\lim_{n \rightarrow \infty} \{(\hat{\rho}(n)|\hat{O}) - (\hat{\rho}(n)|\hat{I})(\hat{\rho}_*|\hat{O})\} = 0 \quad (66)$$

and since $(\hat{\rho}(n)|\hat{I}) = \text{Tr}(\hat{\rho}(n)) = \text{Tr}(\hat{\rho}(0)) = 1$ (conservation of trace given by equation (20)) we have

$$\lim_{n \rightarrow \infty} \{(\hat{\rho}(n)|\hat{O}) - (\hat{\rho}_*|\hat{O})\} = 0 \quad (67)$$

Then,

$$\lim_{n \rightarrow \infty} (\hat{\rho}(n)|\hat{O}) = (\hat{\rho}_*|\hat{O}) \quad (68)$$

which is identical to the limit

$$\lim_{t \rightarrow \infty} (\hat{\rho}(t)|\hat{O}) = (\hat{\rho}_*|\hat{O}) \quad (69)$$

That is, $\hat{\rho}(t)$ weakly converges to $\hat{\rho}_*$ corresponding to the mixing case. Therefore, the quantum Kolmogorov level implies the quantum mixing level.

6 Bernoulli Systems

Essentially the Bernoulli system satisfy the mixing conditions but with no $\lim_{t \rightarrow \infty}$ (see e.g. eqs. (6) and (8)). Then these systems satisfy the following equations:

According to paper [2] eq. (BE) the system is *uniform Bernoulli* if

$$\mu(T_n B \cap A) = \mu(A)\mu(B) \quad (70)$$

or

$$C(T_n B, A) = 0 \quad (71)$$

or for the distribution of density function case (see also the corresponding theorem in [5])

$$\langle P_t f, g \rangle = \langle f_*, g \rangle$$

or in the quantum case, it is quantum Bernoulli if

$$(\hat{\rho}(t)|\hat{O}) = (\hat{\rho}_*|\hat{O}). \quad (72)$$

6.1 Independent Events

Let be $A \subseteq X$ a subset of the phase space. If we interpret $\mu(A)$ as the probability $P(A)$ of that A occurs, then Bernoulli systems satisfy a property expressing the independence between two events of the phase space. This property follows directly from its definition. Let A and B be two subsets belonging to the phase space, then if $n = 0$ in the equation (70) we have the *independence events property*:

$$\mu(B \cap A) = \mu(A)\mu(B) \quad (73)$$

That is, the probability of A and B occur simultaneously is the product of the probability of A by the probability of B .

Now if we take $\mu_* = \mu$ with $\mu_*(A) = \int_A f_*(\phi) d\phi$ then

$$\begin{aligned} \mu_*(B \cap A) &= \mu_*(A)\mu_*(B) \\ \int_{A \cap B} f_*(\phi) d\phi &= \int_A f_*(\phi) d\phi \int_B f_*(\phi) d\phi \end{aligned} \quad (74)$$

Namely,

$$\begin{aligned} \int_X f_* 1_A 1_B d\phi &= \int_X f_* 1_A d\phi \int_X f_* 1_B d\phi \\ \langle f_*, 1_A 1_B \rangle &= \langle f_*, 1_A \rangle \langle f_*, 1_B \rangle \end{aligned} \quad (75)$$

Let be $g_1 = \sum_k a_k 1_{A_k}$, $g_2 = \sum_l b_l 1_{B_l}$. From (75) we have

$$a_k b_l \langle f_*, 1_{A_k} 1_{B_l} \rangle = a_k b_l \langle f_*, 1_{A_k} \rangle \langle f_*, 1_{B_l} \rangle \quad (76)$$

By the linearity of the inner product and summing over the indices k and l we have

$$\langle f_*, \sum_k a_k 1_{A_k} \sum_l b_l 1_{B_l} \rangle = \langle f_*, \sum_k a_k 1_{A_k} \rangle \langle f_*, \sum_l b_l 1_{B_l} \rangle \quad (77)$$

That is,

$$\langle f_*, g_1 g_2 \rangle = \langle f_*, g_1 \rangle \langle f_*, g_2 \rangle \quad (78)$$

Therefore, if $f_* = \text{symb}(\hat{\rho}_*)$ and $g_1 = \text{symb}(\hat{g}_1)$, $g_2 = \text{symb}(\hat{g}_2)$ where \hat{g}_1 and \hat{g}_2 are observables we have

$$\langle \text{symb}(\hat{\rho}_*), g_1 g_2 \rangle = \langle \text{symb}(\hat{\rho}_*), \text{symb}(\hat{g}_1) \rangle \langle \text{symb}(\hat{\rho}_*), \text{symb}(\hat{g}_2) \rangle \quad (79)$$

Now we use that $g_1(\phi)g_2(\phi) \rightarrow \text{symb}(\hat{g}_1\hat{g}_2)$ when $\hbar \rightarrow 0$ (see equation (54)) to obtain

$$\langle \text{symb}(\hat{\rho}_*), \text{symb}(\hat{g}_1\hat{g}_2) \rangle = \langle \text{symb}(\hat{\rho}_*), \text{symb}(\hat{g}_1) \rangle \langle \text{symb}(\hat{\rho}_*), \text{symb}(\hat{g}_2) \rangle \quad (80)$$

Namely,

$$(\hat{\rho}_* | \hat{g}_1 \hat{g}_2) = (\hat{\rho}_* | \hat{g}_1) (\hat{\rho}_* | \hat{g}_2) \quad (81)$$

where in (81) we have used the fundamental property of the *symb* given by the equation (24) of [1]. Moreover, we know that

$$\begin{aligned}
(\hat{\rho}(t)|\hat{g}_1\hat{g}_2) &= (\hat{\rho}_*|\hat{g}_1\hat{g}_2) \\
(\hat{\rho}(t)|\hat{g}_1) &= (\hat{\rho}_*|\hat{g}_1) \\
(\hat{\rho}(t)|\hat{g}_2) &= (\hat{\rho}_*|\hat{g}_2)
\end{aligned} \tag{82}$$

Therefore we can express (81) as

$$(\hat{\rho}(t)|\hat{g}_1\hat{g}_2) = (\hat{\rho}(t)|\hat{g}_1)(\hat{\rho}(t)|\hat{g}_2) \tag{83}$$

for all pairwise of observables \hat{g}_1, \hat{g}_2 . If we generalize for an arbitrary product of observables from (83) it follows that

$$(\hat{\rho}(t)|\prod_i \hat{g}_i) = \prod_i (\hat{\rho}(t)|\hat{g}_i) \tag{84}$$

The equation (84) is the translation into quantum language of the independence of events expressed by the equation (73). Physically, it tells us that in the classical limit of a Bernoulli system the mean value of an arbitrary product of observables factorizes into the product of the mean values of each observable and this factorization occurs for all time.

6.2 Particular Case: Kolmogorov

Bernoulli level is included in Kolmogorov level (equation (70) implies equation (36)) and therefore this property must be verified by the respective quantum versions of these levels. We consider a numerable set of observables $\hat{O}_1, \hat{O}_2, \hat{O}_3, \dots$ and a sequence $m_1, m_2, m_3, \dots \in \mathbb{N}_0$. To demonstrate that quantum Bernoulli level implies quantum Kolmogorov level we use the quantum version of the *independence events property* given by the equation (84). If we call $\hat{g}_1 = \hat{O}_1$, $\hat{g}_2 = \prod_{j=2}^{\infty} \hat{O}_j(n + m_j)$ for all $j = 2, 3, 4, \dots$ by the equation (84) we have

$$(\hat{\rho}(n + m_1)|\hat{O}_1 \prod_{j=2}^{\infty} \hat{O}_j(n + m_j)) = (\hat{\rho}(n + m_1)|\hat{O}_1) \prod_{j=2}^{\infty} (\hat{\rho}(n + m_1)|\hat{O}_j(n + m_j)) \tag{85}$$

In particular since the system is Bernoulli we have

$$(\hat{\rho}(n + m_1)|\hat{O}_1) = (\hat{\rho}_*|\hat{O}_1) \tag{86}$$

From equations (85) and (86) it follows that

$$(\hat{\rho}(n + m_1)|\hat{O}_1 \prod_{j=2}^{\infty} \hat{O}_j(n + m_j)) = \prod_{j=2}^{\infty} (\hat{\rho}(n + m_1)|\hat{O}_j(n + m_j)) (\hat{\rho}_*|\hat{O}_1) \tag{87}$$

Therefore,

$$\lim_{n \rightarrow \infty} (\hat{\rho}(n + m_1)|\hat{O}_1 \prod_{j=2}^{\infty} \hat{O}_j(n + m_j)) = \lim_{n \rightarrow \infty} \prod_{j=2}^{\infty} (\hat{\rho}(n + m_1)|\hat{O}_j(n + m_j)) (\hat{\rho}_*|\hat{O}_1) \tag{88}$$

That is,

$$\lim_{n \rightarrow \infty} \{(\hat{\rho}(n + m_1)|\hat{O}_1 \prod_{j=2}^{\infty} \hat{O}_j(n + m_j)) - \prod_{j=2}^{\infty} (\hat{\rho}(n + m_1)|\hat{O}_j(n + m_j))(\hat{\rho}_*|\hat{O}_1)\} = 0 \quad (89)$$

which is the quantum Kolmogorov condition (see equation (65)).

7 Conclusions

In this paper we have introduced a definition of the four main levels of the Quantum ergodic hierarchy with the property that their classical limits are the corresponding usual levels of the classical ergodic hierarchy.

Language translation of the sigma algebras of the Kolmogorov level to quantum language could be made and reduced to a single condition (see equation (65)) thanks to the application of the principle of inclusion-exclusion (see equation (40)), which helped to extend the technique used in the paper [1] for the ergodic level and mixing level. Here we used the properties of the Wigner transform. The resulting condition for the Kolmogorov quantum level is consistent with the definitions of mixing and Bernoulli (see 5.2 and 6.2 sections). Language translation of the Bernoulli level was the most immediate of all levels of the hierarchy ergodic. However, additionally we have translated the *independence events property* of Bernoulli systems (see equation (73)) into a quantum version in the sense of the expectation values (see equation (84)). The physical interpretation of this property is that the factorization of the expectation value of an observable product in the product of the expectation values of the observables involved. This property was necessary to demonstrate the inclusion of quantum Bernoulli level within the quantum Kolmogorov level (see section 6.2). We have just translated the four levels of the ergodic hierarchy to quantum language, but a great deal of work must be done to find other examples that prove that we are in the good road to understand the notion of quantum chaos. In the next table we list in a compact way the fundamental properties of the Quantum Ergodic Hierarchy levels.

TABLE II (*QUANTUM UNIFORM ERGODIC HIERARCHY*)

LEVEL	CONDITION EQUATION	PROPERTIES
<i>Ergodic</i>	$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\hat{\rho}(t) \hat{O})dt = (\hat{\rho}_* \hat{O})$	Cesaro limit equals to $\hat{\rho}_*$
<i>Mixing</i>	$\lim_{t \rightarrow \infty} (\hat{\rho}(t) \hat{O}) = (\hat{\rho}_* \hat{O})$	Weak limit equals to $\hat{\rho}_*$
<i>Kolmogorov</i>	$\lim_{n \rightarrow \infty} \{(\hat{\rho}(n + m_1) \hat{O}_1 \prod_{j=2}^{\infty} \hat{O}_j(n + m_j)) - \prod_{j=2}^{\infty} (\hat{\rho}(n + m_1) \hat{O}_j(n + m_j))(\hat{\rho}_* \hat{O}_1)\} = 0$	Weak limit equals to $\hat{\rho}_*$
<i>Bernoulli</i>	$(\hat{\rho}(t) \hat{O}) = (\hat{\rho}_* \hat{O})$	$(\hat{\rho}(t) \prod_i \hat{g}_i) = \prod_i (\hat{\rho}(t) \hat{g}_i)$

In this table we can see the level of complexity of the condition that defines each level of the hierarchy ergodic. Starting at the lowest level, the ergodic, which translates into an average temporal of expectation values following by the mixing level corresponding to weak limit. And continuing with Kolmogorov level that represents a condition on a set of observables (the language translation of the sigma algebra) and ending with the Bernoulli level representing the null correlation for all time.

In section 7 of [1] we have presented an example that shows the relevance of the quantum ergodic hierarchy (based on [10]). Moreover in section 4 of [2] the reader can find a long list of classical chaotic systems that can be included in the classic ergodic hierarchy, that now can be also included in the quantum one. The next step in our research is to find physical interesting quantum chaotic models that would belong to the corresponding chaotic hierarchy. This could be the object of forthcoming papers. We believe that this project is feasible and we will try to find a larger set of examples in the future.

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